

Long-Time-Tail Effects on Lyapunov Exponents of a Random, Two-Dimensional Field-Driven Lorentz Gas

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We study the Lyapunov exponents for a moving, charged particle in a two-dimensional Lorentz gas with randomly placed, nonoverlapping hard-disk scatterers in a thermostatted electric field, \vec{E} . The low-density values of the Lyapunov exponents have been calculated with the use of an extended Lorentz-Boltzmann equation. In this paper we develop a method to extend these results to higher density, using the BBGKY hierarchy equations and extending them to include the additional variables needed for calculation of the Lyapunov exponents. We then consider the effects of correlated collision sequences, due to the so-called ring events, on the Lyapunov exponents. For small values of the applied electric field, the ring terms lead to nonanalytic, field-dependent contributions to both the positive and negative Lyapunov exponents which are of the form $\tilde{\epsilon}^2 \ln \tilde{\epsilon}$, where $\tilde{\epsilon}$ is a dimensionless parameter proportional to the strength of the applied field. We show that these nonanalytic terms can be understood as resulting from the change in the collision frequency from its equilibrium value due to the presence of the thermostatted field, and that the collision frequency also contains such nonanalytic terms.

KEY WORDS: Lyapunov exponents; Lorentz gas; extended Lorentz-Boltzmann equation; BBGKY hierarchy equations; long-time-tail effect.

1. INTRODUCTION

The Lorentz gas has proved to be a useful model for studying the relations between dynamical systems theory and nonequilibrium properties of many body systems. This model consists of a set of scatterers that are fixed in

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space together with moving particles that collide with the scatterers. Here we consider the version of the model in two dimensions where the scatterers are fixed hard disks, placed at random in the plane without overlapping. Each of the moving particles is a point particle with a mass and a charge, and is subjected to an external, uniform electric field as well as a Gaussian thermostat which is designed to keep the kinetic energy of the moving particle at a constant value. The particles make elastic, specular collisions with the scatterers, but do not interact with each other. The interest in the Lorentz gas model stems from the fact that its chaotic properties can be analyzed in some detail, at least if the scatterers form a sufficiently dilute, quenched gas, so that the average distance between scatterers is large compared to their radii. The interest in a thermostatted electric field arises from the fact that at small fields a transport coefficient, the electrical conductivity of the particles, is proportional to the sum of the Lyapunov exponents describing the chaotic motion of the moving particles.⁽¹⁾ The Lyapunov exponents are to be calculated for the case where the charged particle is described by a nonequilibrium steady state phase-space distribution function which is reached from some typical initial distribution function after a sufficiently long period of time. In this state, the distribution function for an ensemble of moving particles (all interacting with the scatterers and the field, but not with each other) is independent of time and its average over the distribution of scatterers is spatially homogeneous. It is known from computer simulations^(2,3) and theoretical discussions^(4,5) that in the stationary state the trajectories of the moving particles in phase space lie on a fractal attractor of lower dimension than the dimension of the constant energy surface, which is three dimensional for the constant energy Lorentz gas in two dimensions. There can be at most two non-zero Lyapunov exponents for this model since the Lyapunov exponent in the direction of the phase-space trajectory is zero. Also, the relation between the Lyapunov exponents and the electrical conductivity requires that the sum of the non-zero exponents should be negative due to the positivity of electrical conductivity.⁽⁴⁾

The case of the dilute, random Lorentz gas has already been studied in detail. Van Beijeren and coworkers^(6,7) have calculated the Lyapunov spectrum for an equilibrium Lorentz gas in two and three dimensions using various kinetic theory methods including Boltzmann equation techniques. These methods were also applied to the dilute, random Lorentz gas in a thermostatted electric field with results for the Lyapunov exponents that are in excellent agreement with computer simulations.^(8,9) Moreover, the results for the field-dependent case were in accord with the relation between the electrical conductivity and the Lyapunov exponents for the moving particle.

The purpose of this paper is to extend the results obtained for the Lyapunov exponents for dilute Lorentz gases to higher densities. Our central themes will be: (a) to describe a general method, based upon the BBGKY hierarchy equations, for accomplishing this task, and (b) to examine the effects on the Lyapunov exponents of long range correlations in time between the moving particle and the scatterers produced by correlated collision sequences where the particle collides with a given scatterer more than once and the time interval between such re-collisions is on the order of several mean free times, with an arbitrary number of intermediate collisions with other scatterers. These correlated collision sequences are of particular interest in kinetic and transport theory because they are responsible for the “long-time-tail” effects in the Green-Kubo time correlation functions, which lead to various divergences in the transport coefficients for two- and three-dimensional gases, where all of the particles move.⁽¹⁰⁾ In the case of a Lorentz gas in d dimensions, the Green-Kubo velocity correlation functions decay with time, t , as $t^{-(d/2+1)}$ ⁽¹¹⁾ and the diffusion coefficient is finite in both two and three dimensions. Here we describe the effects of these type of correlations on the Lyapunov exponents for the two-dimensional Lorentz gas, in equilibrium, where we find no effect, and in a thermostatted electric field, where we find a small, logarithmic dependence of the Lyapunov exponents upon the applied field. This logarithmic effect is an indicator for similar effects to be expected when one calculates Lyapunov exponents associated with more general transport in two-dimensional gases. In the case of the Lorentz gas, at least, the logarithmic terms can easily be associated with the logarithmic terms that appear in the field dependent collision frequency, and a very simple argument can be used to establish this relation between logarithmic terms in the Lyapunov exponents and in the collision frequency.

In Section 2 of this article we describe the general theory of Lyapunov exponents of a two-dimensional thermostatted electric field-driven Lorentz gas and quote the results within the scope of the Boltzmann equation. In Section 3, we generalize the theory to incorporate the effect of correlated collision sequences. In Section 4, we outline the calculation of the effects of the correlated collision sequences on the non-zero Lyapunov exponents using the BBGKY equations discussed in Section 3 and obtain the non-analytic field-dependent term in the Lyapunov exponents, originating from the correlated collision sequences, along with other analytic field-dependent terms. In Section 5, we present some simple arguments explaining the field-dependence of the collision frequency and show that this is the sole origin of the nonanalytic, field-dependent terms in the Lyapunov exponents. Notice that the arguments given in Section 5 are independent of and much

simpler to follow than the formalism developed in Sections 3 and 4. We conclude in Section 6 with a discussion of the results obtained here, and with a consideration of open questions. Methods for determination of the field-dependence of the collision frequency are outlined in the Appendix.

2. LYAPUNOV EXPONENTS OF FIELD DRIVEN LORENTZ GASES IN TWO DIMENSIONS

2.1. General Theory

The random Lorentz gas consists of point particles of mass m and charge q moving in a random array of fixed scatterers. In two dimensions, each scatterer is a hard disk of radius a . The disks do not overlap with each other and are distributed with number density n , such that at low density $na^2 \ll 1$. The point particles are acted upon by a uniform, constant electric field \vec{E} in the \hat{x} direction, but there is no interaction between any two point particles. There is also a Gaussian thermostat in the system to keep the speed of each particle constant at v by means of a dynamical friction during flights between collisions with the scatterers. The collisions between a point particle and the scatterers are instantaneous, specular and elastic. During a flight, the equations of motion of a point particle are

$$\dot{\vec{r}} = \vec{v} = \frac{\vec{p}}{m}, \quad \dot{\vec{p}} = m\dot{\vec{v}} = q\vec{E} - \alpha\vec{p} \quad (1)$$

and at a collision with a scatterer, the post-collisional position and velocity, \vec{r}_+ and \vec{v}_+ , are related to the pre-collisional position and velocity, \vec{r}_- and \vec{v}_- , by

$$\vec{r}_+ = \vec{r}_-, \quad \vec{v}_+ = \vec{v}_- - 2(\vec{v}_- \cdot \hat{\sigma}) \hat{\sigma}, \quad (2)$$

where $\hat{\sigma}$ is the unit vector from the center of the scatterer to the point of collision (see Fig. 1). The fact that each particle has a constant speed v determines the value of α :

$$\alpha = \frac{q\vec{E} \cdot \vec{p}}{p^2} \Rightarrow \dot{\vec{p}} = q\vec{E} - \frac{q\vec{E} \cdot \vec{p}}{p^2} \vec{p}. \quad (3)$$

Equivalently, in polar coordinates, the velocity direction with respect to the field, defined through $\hat{v} \cdot \hat{x} = \cos \theta$, changes between collisions as

$$\dot{\theta} = -\varepsilon \sin \theta, \quad (4)$$

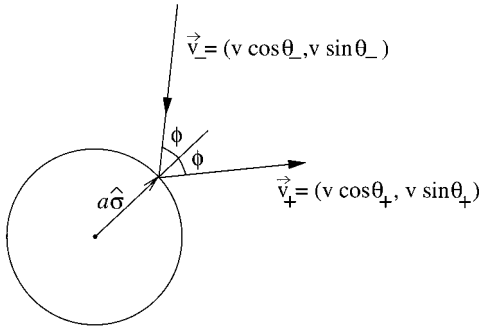


Fig. 1. Collision between a point particle and a scatterer.

where $\varepsilon = q |\vec{E}|/(mv)$ and we define the dimensionless electric field parameter $\tilde{\varepsilon} = \varepsilon l/v$, where $l = (2na)^{-1}$ is the mean free path length for the particle in the dilute Lorentz gas. To denote the electric field, we will normally use ε , though from time to time we will use $\tilde{\varepsilon}$, too.

Treating this two-dimensional Lorentz gas as a dynamical system, we define the Lyapunov exponents in the usual way: a point particle in its phase space $(\vec{r}, \vec{v}) = \vec{X}$ starts at time t_0 at a phase space location $\vec{X}(t_0)$. Under time evolution, $\vec{X}(t)$ follows a trajectory in this phase space which we call the “reference trajectory”. We consider an infinitesimally displaced trajectory which starts at the same time t_0 , but at $\vec{X}'(t_0) = \vec{X}(t_0) + \delta\vec{X}(t_0)$. Under time evolution, $\vec{X}'(t)$ follows another trajectory, always staying infinitesimally close to the reference trajectory. This trajectory we call the “adjacent trajectory”. Typically the two trajectories will separate in time due to the convex nature of the collisions. Thus, we can define the positive Lyapunov exponent as

$$\lambda_+ = \lim_{T \rightarrow \infty} \lim_{|\delta\vec{X}(t_0)| \rightarrow 0} \frac{1}{T} \ln \frac{|\delta\vec{X}(t_0 + T)|}{|\delta\vec{X}(t_0)|}, \tag{5}$$

for a typical trajectory of the system.

We assume that, for small fields, this Lorentz gas system is hyperbolic. Since the two-dimensional Lorentz gas can have at most two nonzero Lyapunov exponents, we denote the negative Lyapunov exponent by λ_- . Without any loss of generality, we can choose to measure the separation of the reference and adjacent trajectories equivalently in \vec{r} -space, thereby reducing the definition of the positive Lyapunov exponent to

$$\lambda_+ = \lim_{T \rightarrow \infty} \lim_{|\delta\vec{r}(t_0)| \rightarrow 0} \frac{1}{T} \ln \frac{|\delta\vec{r}(t_0 + T)|}{|\delta\vec{r}(t_0)|}. \tag{6}$$

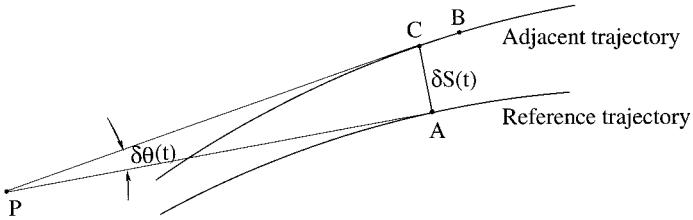


Fig. 2. $\rho(t) = \delta S(t) / \delta \theta(t) = |AP|$.

In order to calculate the right hand side of Eq. (6), we introduce another dynamical quantity, the radius of curvature ρ , characterizing the spatial separation of the two trajectories (see Fig. 2). In Fig. 2, a particle on the reference trajectory would be at point A at time t . At the same time, a particle on the adjacent trajectory would be at B. A local perpendicular on the reference trajectory at A intersects the adjacent trajectory at C. The backward extensions of instantaneous velocity directions on the reference and adjacent trajectories at A and C, respectively, intersect each other at point P. We denote the length of the line segment AC by $\delta S(t)$ and $\angle APC$ by $\delta \theta(t)$. The radius of curvature associated with the particle on the reference trajectory at time t is then given by

$$\rho(t) = \frac{\delta S(t)}{\delta \theta(t)} = |AP|. \quad (7)$$

Having defined $\rho(t)$, one can make a simple geometric argument to show that

$$\delta \dot{S}(t) = v \delta \theta(t) = \frac{v \delta S(t)}{\rho(t)}, \quad (8)$$

so as to obtain a version of Sinai's formula,⁽¹²⁾

$$\lambda = \lim_{T \rightarrow \infty} \frac{v}{T} \int_{t_0}^{t_0+T} \frac{dt}{\rho(t)}. \quad (9)$$

During a flight, the equation of motion for ρ is given by⁽⁸⁾

$$\dot{\rho} = v + \rho \varepsilon \cos \theta + \frac{\rho^2 \varepsilon^2 \sin^2 \theta}{v}. \quad (10)$$

At a collision with a scatterer, the post-collisional velocity angle θ_+ and radius of curvature ρ_+ are related to the pre-collisional velocity angle θ_- and radius of curvature ρ_- by:^(13, 14)

$$\theta_+ = \theta_- - \pi + 2\phi, \quad \frac{1}{\rho_+} = \frac{1}{\rho_-} + \frac{2}{a \cos \phi} + \frac{\varepsilon}{v} \tan \phi (\sin \theta_- + \sin \theta_+), \quad (11)$$

where ϕ is the collision angle, i.e., $\cos \phi = |\hat{v}_- \cdot \hat{\sigma}| = |\hat{v}_+ \cdot \hat{\sigma}|$ (see Fig. 1).

Now we assume that, for sufficiently weak electric field, the field-driven Lorentz gas in two dimensions is ergodic, and that we can replace the long time average in Eq. (6) by a non-equilibrium steady state (NESS) average, including an average over all allowed configurations of scatterers, to obtain

$$\lambda = \left\langle \frac{v}{\rho} \right\rangle_{\text{NESS}}. \quad (12)$$

The electric field is considered weak if the work done by the electric field on the point particle over a flight of one mean free path is much smaller than the particle's kinetic energy, i.e., $q |\vec{E}| l / (mv^2) = \varepsilon l / v = \tilde{\varepsilon} \ll 1$.

We note for future reference, that the sum of the two nonzero Lyapunov exponents is related to the average of the friction coefficient α , by^(1, 15, 16)

$$\lambda_+ + \lambda_- = -\langle \alpha \rangle_{\text{NESS}} = -\left\langle \frac{q\vec{E} \cdot \vec{v}}{mv^2} \right\rangle_{\text{NESS}} = -\frac{\vec{J} \cdot \vec{E}}{mv^2} = -\frac{\sigma E^2}{mv^2}. \quad (13)$$

Here the electric current $\vec{J} = \langle q\vec{v} \rangle_{\text{NESS}}$ is, for small fields, assumed to satisfy Ohm's law, $\vec{J} = \sigma \vec{E}$, and σ is the electrical conductivity.

2.2. Results Obtained Using the Lorentz–Boltzmann Equation

To the lowest order in density, one can assume that the collisions suffered by the point particle are uncorrelated, and use an extended Lorentz–Boltzmann equation (ELBE) for the distribution function of the moving particle, $f_1(\vec{r}, \vec{v}, \rho, t)$ in (\vec{r}, \vec{v}, ρ) -space⁽⁸⁾ needed for the evaluation of the averages appearing in Eqs. (12) and (13). To calculate the positive Lyapunov exponent, one needs to consider the forward-time ELBE while to calculate the negative Lyapunov exponent one needs the time reversed ELBE. To the leading order in density, the Lyapunov exponents are then given by:⁽⁸⁾

$$\lambda_+^{(B)} = \lambda_0 - \frac{11}{48} \frac{l}{v} \varepsilon^2 + O(\varepsilon^4) \quad \text{and} \quad \lambda_-^{(B)} = -\lambda_0 - \frac{7}{48} \frac{l}{v} \varepsilon^2 + O(\varepsilon^4). \quad (14)$$

The superscript, B, indicates that these are results obtained from the Lorentz–Boltzmann equation. Here λ_0 is the positive Lyapunov exponent for a field-free Lorentz gas (see for example, Ref. 17) given by

$$\lambda_0 = 2nav[1 - \mathcal{C} - \ln(2na^2)], \quad (15)$$

where \mathcal{C} is Euler's constant, $\mathcal{C} = 0.5772\dots$. From Eqs. (13) and (14), using Einstein's relation between diffusion coefficient and conductivity, one gets the correct diffusion coefficient within the Boltzmann regime, $D^{(B)} = \frac{3}{8}lv$.

To derive the results in Eq. (14), one uses Eq. (11) with $\varepsilon = 0$. The ε -dependent term in Eq. (11) can be explicitly shown to be of higher order in the density than the terms present in Eq. (14).⁽¹⁸⁾ In the following sections we will investigate the effect of sequences of correlated collisions between the point particle and the scatterers on the Lyapunov exponents. However, the ε -dependent term in Eq. (11) will again be neglected since we will present the effect of these correlated collision sequences in leading order in the density of scatterers only. Thus, instead of Eq. (11), we will use

$$\frac{1}{\rho_+} = \frac{1}{\rho_-} + \frac{2}{a \cos \phi}. \quad (16)$$

3. THE EXTENSION OF THE ELBE TO HIGHER DENSITY

3.1. Binary Collision Operators in (\vec{r}, \vec{v}, ρ) -Space and the BBGKY Hierarchy Equations

The Boltzmann theory for the Lyapunov exponents assumes that the scatterers form a dilute, but quenched system and that the collisions of the point particles with the scatterers are uncorrelated. To incorporate the effects of correlated collisions on the Lyapunov exponents, we will use a method based on the BBGKY hierarchy equations, familiar from the kinetic theory of moderately dense gases.⁽¹⁰⁾ Since the moving particles do not interact with each other, it is sufficient to consider the distribution functions for just one of them, together with a number of scatterers. One starts from a fundamental equation for an $(N+1)$ -body distribution function, $f_{N+1} = f_{N+1}(\vec{r}, \vec{v}, \rho; \vec{R}_1, \vec{R}_2, \dots, \vec{R}_N; t)$, which is the probability density function in the entire extended phase space Γ spanned by the variables $\vec{r}, \vec{v}, \rho, \vec{R}_1, \vec{R}_2, \dots, \vec{R}_N$, describing our system of N scatterers and one moving particle. We require that f_{N+1} satisfies the normalization condition

$$\int d\vec{r} d\vec{v} d\rho d\vec{R}_1 d\vec{R}_2 \dots d\vec{R}_N f_{N+1}(\vec{r}, \vec{v}, \rho; \vec{R}_1, \vec{R}_2, \dots, \vec{R}_N; t) = 1. \quad (17)$$

This $(N + 1)$ -body distribution function satisfies a Liouville-like equation determined by the collisions of the moving particles with the scatterers and by the motion of the particles in the thermostatted electric field, between collisions. Since the time evolution of \vec{r} and \vec{v} in this field is not Hamiltonian, we must use the Liouville equation in the form of a conservation law, rather than the usual form for Hamiltonian systems, to obtain

$$\frac{\partial f_{N+1}}{\partial t} + \vec{\nabla}_{\vec{r}} \cdot (\dot{\vec{r}} f_{N+1}) + \vec{\nabla}_{\vec{v}} \cdot (\dot{\vec{v}} f_{N+1}) + \frac{\partial}{\partial \rho} (\dot{\rho} f_{N+1}) = \sum_{i=1}^N \tilde{T}_{-,i} f_{N+1}. \quad (18)$$

Here the operators $\tilde{T}_{-,i}$ are binary collision operators which describe the effects on the distribution function due to an instantaneous, elastic collision between the moving particle and the scatterer labeled by the index i . The explicit form of the binary collision operators may be easily obtained by a slight modification of the methods used by Ernst *et al.*,⁽¹⁹⁾ in order to include the radius of curvature as an additional variable. One finds that the action of this operator on any function $f(\vec{r}, \vec{v}, \rho; \vec{R}_1, \vec{R}_2, \dots, \vec{R}_j; t)$ is

$$\begin{aligned} \tilde{T}_{-,i} f = & a \int_{\vec{v} \cdot \hat{\sigma}_i > 0} d\hat{\sigma}_i |\vec{v} \cdot \hat{\sigma}_i| \left\{ \int_0^\infty d\rho' \delta \left(\rho - \frac{\rho' a \cos \phi_i}{a \cos \phi_i + 2\rho'} \right) \delta(a\hat{\sigma}_i - (\vec{r} - \vec{R}_i)) \right. \\ & \left. \times b_{\sigma_i, \rho'} - \delta(a\hat{\sigma}_i + (\vec{r} - \vec{R}_i)) \right\} f, \end{aligned} \quad (19)$$

where $\hat{\sigma}_i$ is the unit vector from the center of the scatterer fixed at \vec{R}_i to the point of collision. The action of the operator $b_{\sigma_i, \rho'}$ on the function $f(\vec{r}, \vec{v}, \rho; \vec{R}_1, \vec{R}_2, \dots, \vec{R}_j; t)$ is defined by

$$b_{\sigma_i, \rho'} f(\vec{r}, \vec{v}, \rho; \vec{R}_1, \vec{R}_2, \dots, \vec{R}_j; t) = f(\vec{r}, \vec{v} - 2(\vec{v} \cdot \hat{\sigma}_i) \hat{\sigma}_i, \rho'; \vec{R}_1, \vec{R}_2, \dots, \vec{R}_j; t); \quad (20)$$

that is, $b_{\sigma_i, \rho'}$ is a substitution operator that replaces ρ by ρ' and the velocity \vec{v} by its restituting value, i.e., the value it should have before collision so as to lead to the value \vec{v} after collision. It is often useful to express the binary collision operators as sum of two terms such that

$$\tilde{T}_{-,i} = \tilde{T}_{-,i}^{(+)} - \tilde{T}_{-,i}^{(-)}, \quad (21)$$

where

$$\begin{aligned} \tilde{T}_{-,i}^{(+)} = & a \int_{\vec{v} \cdot \hat{\sigma}_i > 0} d\hat{\sigma}_i |\vec{v} \cdot \hat{\sigma}_i| \int_0^\infty d\rho' \delta \left(\rho - \frac{\rho' a \cos \phi_i}{a \cos \phi_i + 2\rho'} \right) \\ & \times \delta(a\hat{\sigma}_i - (\vec{r} - \vec{R}_i)) b_{\sigma_i, \rho'}, \end{aligned} \quad (22)$$

and

$$\tilde{T}_{-,i}^{(-)} = a \int_{\vec{v} \cdot \hat{\sigma}_i > 0} d\hat{\sigma}_i |\vec{v} \cdot \hat{\sigma}_i| \delta(a\hat{\sigma}_i + (\vec{r} - \vec{R}_i)). \quad (23)$$

One sees that $\tilde{T}_{-,i}^{(+)} f$ and $\tilde{T}_{-,i}^{(-)} f$ respectively describe the rate of “gain” and the rate of “loss” of f due to a collision of the point particle with the scatterer fixed at \vec{R}_i .

The BBGKY hierarchy equations are then obtained from Eq. (18) by integrating over scatterer coordinates, as a set of equations for the reduced distributions f_j for the moving particle and $(j-1)$ scatterers, defined by

$$\begin{aligned} f_j(\vec{r}, \vec{v}, \rho; \vec{R}_1, \vec{R}_2, \dots, \vec{R}_{j-1}; t) \\ = \frac{N!}{(N-j+1)!} \int d\vec{R}_j \dots d\vec{R}_N f_{N+1}(\vec{r}, \vec{v}, \rho; \vec{R}_1, \vec{R}_2, \dots, \vec{R}_N; t). \end{aligned} \quad (24)$$

One then easily obtains the BBGKY hierarchy equations ($1 \leq j \leq N$)

$$\frac{\partial f_j}{\partial t} + \vec{\nabla}_{\vec{r}} \cdot (\dot{\vec{r}} f_j) + \vec{\nabla}_{\vec{v}} \cdot (\dot{\vec{v}} f_j) + \frac{\partial}{\partial \rho} (\dot{\rho} f_j) - \sum_{k=1}^{j-1} \tilde{T}_{-,k} f_j = \int d\vec{R}_j \tilde{T}_{-,j} f_{j+1}. \quad (25)$$

3.2. Cluster Expansions and Truncation of the Hierarchy Equations

The usual procedure for truncating the hierarchy equations in order to obtain the Boltzmann equation and its extension to higher densities is to make cluster expansions of the distribution functions, f_2, f_3, \dots in terms of a set of correlation functions, g_2, g_3, \dots as follows:

$$f_2(\vec{r}, \vec{v}, \rho; \vec{R}_1; t) = n f_1(\vec{r}, \vec{v}, \rho; t) + g_2(\vec{r}, \vec{v}, \rho; \vec{R}_1; t) \quad (26)$$

$$\begin{aligned} f_3(\vec{r}, \vec{v}, \rho; \vec{R}_1, \vec{R}_2; t) = n^2 f_1(\vec{r}, \vec{v}, \rho; t) + n g_2(\vec{r}, \vec{v}, \rho; \vec{R}_1; t) \\ + n g_2(\vec{r}, \vec{v}, \rho; \vec{R}_2; t) + g_3(\vec{r}, \vec{v}, \rho; \vec{R}_1, \vec{R}_2; t), \end{aligned} \quad (27)$$

and so on. Hereafter, to save writing, we denote $g_2(\vec{r}, \vec{v}, \rho; \vec{R}_1; t)$ as g_{2, \vec{R}_1} , $g_2(\vec{r}, \vec{v}, \rho; \vec{R}_2; t)$ as g_{2, \vec{R}_2} , $f_3(\vec{r}, \vec{v}, \rho; \vec{R}_1, \vec{R}_2; t)$ as f_3 and $g_3(\vec{r}, \vec{v}, \rho; \vec{R}_1, \vec{R}_2; t)$ as g_3 . The first terms in each of these expansions represent the totally uncorrelated situation, where there are independent probabilities of finding the moving particle and the scatterers at the designated coordinates. The next terms involving the pair correlation functions g_{2, \vec{R}_i} in Eqs. (26) and (27)

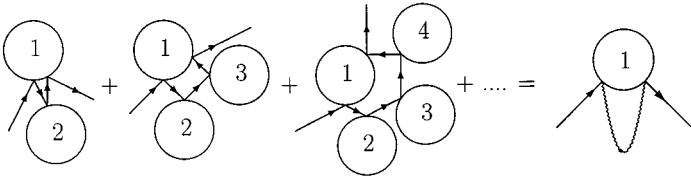


Fig. 3. Sequential collisions with scatterers at $\vec{R}_2, \vec{R}_3, \dots$ adding up to the ring diagram.

take into account possible dynamical and excluded volume correlations between the point particle and the scatterer at \vec{R}_i . If one replaces f_2 by nf_1 in the first BBGKY hierarchy equation, Eq. (25) with $j=1$, reduces to the ELBE. To find the corrections to the ELBE for higher densities, one must keep the g_2 term in Eq. (26) and use the second hierarchy equation to determine g_2 . However, in order to solve the second equation, we have to say something about g_3 . A careful examination of the second and higher equations shows that g_3 contains, of course, the effects of three-body correlations, i.e., correlated collisions involving the point particle, a scatterer fixed at \vec{R}_1 and another scatterer fixed at \vec{R}_2 , as well as excluded volume corrections due to the nonoverlapping property of the scatterers. Here we will be primarily interested in the effects of the so called “ring” collisions on the Lyapunov exponents. These collision sequences are composed of one collision of the moving particle with a given scatterer, followed by an arbitrary number of collisions with a succession of different scatterers, and completed by a final re-collision of the moving particle with the first scatterer in the sequence, as illustrated in Fig. 3.

The ring diagrams, taken individually, are the most divergent terms that appear in the expansion of dynamical properties of the Lorentz gas as a *power* series in the density of scatterers. They lead to the logarithmic terms in the density expansion of the diffusion coefficient of the moving particle,⁽²⁰⁾ and to the algebraic long time tails in the velocity time correlation function of the moving particle.⁽¹¹⁾ While many other dynamical events and excluded volume effects contribute to the Lyapunov exponents, and must be included for a full treatment, we concentrate here on the effects of these most divergent collision sequences, since in other contexts, they are responsible for the most dramatic higher density corrections to the Boltzmann equation results.

Thus we drop g_3 in Eq. (27) and obtain a somewhat simplified cluster expansion of f_3 , given by

$$f_3 = n^2 f_1 + n g_{2, \vec{R}_1} + n g_{2, \vec{R}_2}. \tag{28}$$

Using Eqs. (26) and (28) and the first two of the BBGKY hierarchy equations, we obtain a closed set of two equations involving two unknowns, f_1 and g_{2, \vec{R}_1} given by

$$\vec{\nabla}_{\vec{v}} \cdot (\dot{v}f_1) + \frac{\partial}{\partial \rho} (\dot{\rho}f_1) = \int d\vec{R}_1 \tilde{T}_{-,1} [nf_1 + g_{2, \vec{R}_1}], \quad (29)$$

and

$$\begin{aligned} \vec{\nabla}_{\vec{r}} \cdot (\dot{r}g_{2, \vec{R}_1}) + \vec{\nabla}_{\vec{v}} \cdot (\dot{v}g_{2, \vec{R}_1}) + \frac{\partial}{\partial \rho} (\dot{\rho}g_{2, \vec{R}_1}) \\ - n \int d\vec{R}_2 \tilde{T}_{-,2} g_{2, \vec{R}_1} = n\tilde{T}_{-,1} f_1. \end{aligned} \quad (30)$$

In the derivation of Eq. (30) from the second hierarchy equation not only have we dropped g_3 as discussed above, we also dropped a term of the form $\tilde{T}_{-,1} g_{2, \vec{R}_1}$. This term provides ‘‘repeated ring’’ corrections to the ring contributions to g_{2, \vec{R}_1} . These are of the same order as terms neglected by dropping g_3 ,^(21, 18) and should be neglected for consistency. We also dropped the time derivatives in the equations, so we are now looking for the distribution and correlation functions appropriate for the NESS.

In Section 4, we will solve Eqs. (29) and (30) in order to calculate the ring contributions to the positive Lyapunov exponent. Before doing so, it is useful to write down the usual form of the ring equations in (\vec{r}, \vec{v}) -space, which can be obtained by integrating Eqs. (29) and (30) over all values of the radius of curvature, $0 \leq \rho < \infty$. We define the usual single-particle distribution function by, $F_1 = \int_{\rho > 0} d\rho f_1$ and the pair-correlation function $G_{2, \vec{R}_1} = \int_{\rho > 0} d\rho g_{2, \vec{R}_1}$. By imposing the boundary conditions that both f_1 and g_{2, \vec{R}_1} go to zero as $\rho \rightarrow 0$ and as $\rho \rightarrow \infty$, we obtain

$$\vec{\nabla}_{\vec{v}} \cdot (\dot{v}F_1) = \int d\vec{R}_1 \bar{T}_{-,1} [nF_1 + G_{2, \vec{R}_1}], \quad (31)$$

and

$$\vec{\nabla}_{\vec{r}} \cdot (\dot{r}G_{2, \vec{R}_1}) + \vec{\nabla}_{\vec{v}} \cdot (\dot{v}G_{2, \vec{R}_1}) - n \int d\vec{R}_2 \bar{T}_{-,2} G_{2, \vec{R}_1} = n\bar{T}_{-,1} F_1. \quad (32)$$

The actions of $\bar{T}_{-,1}$ or $\bar{T}_{-,2}$ on F_1 and G_{2, \vec{R}_1} can be obtained by appropriately integrating $\tilde{T}_{-,1} f_1$, $\tilde{T}_{-,1} g_{2, \vec{R}_1}$ or $\tilde{T}_{-,2} g_{2, \vec{R}_1}$ over ρ from 0 to ∞ using the definitions in Eqs. (19) and (20). $\bar{T}_{-,1}$ and $\bar{T}_{-,2}$ are the analogs in (\vec{r}, \vec{v}) space of $\tilde{T}_{-,1}$ and $\tilde{T}_{-,2}$ (see Eqs. (19) and (20)), i.e.,

$$\bar{T}_{-,i} = a \int_{\vec{v} \cdot \hat{\sigma}_i > 0} d\hat{\sigma}_i |\vec{v} \cdot \hat{\sigma}_i| \{ \delta(a\hat{\sigma}_i - (\vec{r} - \vec{R}_i)) b_{\sigma_i} - \delta(a\hat{\sigma}_i + (\vec{r} - \vec{R}_i)) \}. \quad (33)$$

In future applications however, we will drop the $a\hat{\sigma}_i$ terms from the arguments of both $\delta(a\hat{\sigma}_i \pm (\vec{r} - \vec{R}_i))$ in $\tilde{T}_{-,i}$ and $\bar{T}_{-,i}$ operators since they lead to corrections similar to excluded volume terms, neglected already.

4. EFFECTS OF LONG RANGE TIME CORRELATION ON λ_+ AND λ_-

We now concentrate on the solution of the BBGKY equations for the distribution functions that determine the Lyapunov exponents. The solutions of Eqs. (29) and (30) are to be obtained as expansions in two small variables, na^2 and $\tilde{\varepsilon}$. The density expansion will give the corrections to the previously obtained Boltzmann regime results from the ELBE, and the $\tilde{\varepsilon}$ expansion will provide the field dependence of these corrections. We therefore write the density expansions of f_1 and g_2 (hereafter we drop the subscript \vec{R}_1 from g_{2,\vec{R}_1}) as

$$f_1 = f_1^{(B)} + f_1^{(R)} + \dots, \quad \text{and} \quad g_2 = g_2^{(R)} + \dots, \quad (34)$$

where the superscript B indicates the lowest density result for f_1 as given by the ELBE, and the superscript R denotes the ring contribution. At the order in density of interest here, the quantities indicated explicitly in the above equations satisfy

$$\vec{\nabla}_{\vec{v}} \cdot (\dot{\vec{v}} f_1^{(B)}) + \frac{\partial}{\partial \rho} (\dot{\rho} f_1^{(B)}) = n \int d\vec{R}_1 \tilde{T}_{-,1} f_1^{(B)}, \quad (35)$$

$$\vec{\nabla}_{\vec{v}} \cdot (\dot{\vec{v}} f_1^{(R)}) + \frac{\partial}{\partial \rho} (\dot{\rho} f_1^{(R)}) = \int d\vec{R}_1 \tilde{T}_{-,1} [n f_1^{(R)} + g_2^{(R)}], \quad (36)$$

and

$$\vec{\nabla}_{\vec{r}} \cdot (\dot{\vec{r}} g_2^{(R)}) + \vec{\nabla}_{\vec{v}} \cdot (\dot{\vec{v}} g_2^{(R)}) + \frac{\partial}{\partial \rho} (\dot{\rho} g_2^{(R)}) - n \int d\vec{R}_2 \tilde{T}_{-,2} g_2^{(R)} = n \tilde{T}_{-,1} f_1^{(B)}. \quad (37)$$

Our aim here is to solve Eqs. (36) and (37) using the results of the ELBE for $f_1^{(B)}$. We suppose further that each of these functions possesses an expansion in powers of ε as

$$f_1^{(B,R)} = f_{1,0}^{(B,R)} + \varepsilon f_{1,1}^{(B,R)} + \varepsilon^2 f_{1,2}^{(B,R)} + \dots, \quad (38)$$

and

$$g_2^{(R)} = g_{2,0}^{(R)} + \varepsilon g_{2,1}^{(R)} + \varepsilon^2 g_{2,2}^{(R)} + \dots. \quad (39)$$

The functions $f_{1,i}^{(B)}$ have been previously obtained as the ε solutions of the ELBE. Since we will be dealing with g_2 only in the context of the ring term, we drop the superscript R from now on.

As mentioned above, we will neglect the term $a\hat{\sigma}$ within the argument of the δ functions appearing in each of the binary collision operators \tilde{T}_- and \bar{T}_- , so as to take the moving particle to be located at the same point as the center of the appropriate scatterer at collision. The terms neglected by this approximation lead to higher density corrections to the terms we will obtain below. Secondly, an inspection of the radius of curvature delta function in the expression for the “gain” part of the binary collision operator, Eq. (22), shows that this term is only nonvanishing when $\rho \leq a/2$, and that the dominant contribution to the ρ' integration comes from the region $\rho' \sim l$. Naturally, $\rho'a \cos \phi_i / (a \cos \phi_i + 2\rho') \sim (a \cos \phi_i / 2)(1 + O(n))$ in the argument of the delta function. In the Boltzmann level approximation this $O(n)$ term may therefore be neglected and it can be shown not to contribute to the leading field-dependent ring term effects on the Lyapunov exponents. Therefore, we will neglect it in what follows. Under this approximation, the gain part of the binary collision operator⁽⁸⁾ acts on an arbitrary function $h(\vec{r}, \vec{v}, \rho)$ as

$$\begin{aligned} \tilde{T}_{-,i}^{(+)} h(\vec{r}, \vec{v}, \rho) &\approx \delta(\vec{r} - \vec{R}_i) \Theta\left(\frac{a}{2} - \rho\right) I(\rho) [H(\vec{r}, \vec{v}_+) + H(\vec{r}, \vec{v}_-)] \\ &\equiv \delta(\vec{r} - \vec{R}_i) \Gamma(\rho, H). \end{aligned} \quad (40)$$

Here

$$\vec{v}_{\pm} = \vec{v} - 2(\vec{v} \cdot \hat{\sigma}_{i,\pm}) \hat{\sigma}_{i,\pm}, \quad (41)$$

and $\hat{\sigma}_{i,\pm}$ are defined by the condition that the scattering angle $\phi = \pm \cos^{-1}(2\rho/a)$. Also

$$H(\vec{r}, \vec{v}) = \int_0^{\infty} d\rho' h(\vec{r}, \vec{v}, \rho'), \quad (42)$$

and

$$I(\rho) = \frac{4v\rho}{a \sqrt{1 - \left(\frac{2\rho}{a}\right)^2}}. \quad (43)$$

Finally, we express the velocity vector \vec{v} in terms of the angle θ that it makes with the direction of the electric field so as to obtain the following set of equations for the terms in the ε -expansion of g_2

$$\left[\vec{v} \cdot \vec{\nabla}_{\vec{r}} + v \frac{\partial}{\partial \rho} + 2nav \right] g_{2,0} = -2nav \delta(\vec{r} - \vec{R}_1) f_{1,0}^{(B)} + n \delta(\vec{r} - \vec{R}_1) \Gamma(\rho, F_{1,0}^{(B)}), \quad (44)$$

$$\begin{aligned} \left[\vec{v} \cdot \vec{\nabla}_{\vec{r}} + v \frac{\partial}{\partial \rho} + 2nav \right] g_{2,1} &= -\frac{\partial}{\partial \rho} (\rho \cos \theta g_{2,0}) + \frac{\partial}{\partial \theta} (\sin \theta g_{2,0}) \\ &\quad - 2nav \delta(\vec{r} - \vec{R}_1) f_{1,1}^{(B)} + n \delta(\vec{r} - \vec{R}_1) \Gamma(\rho, F_{1,1}^{(B)}), \end{aligned} \quad (45)$$

and

$$\begin{aligned} \left[\vec{v} \cdot \vec{\nabla}_{\vec{r}} + v \frac{\partial}{\partial \rho} + 2nav \right] g_{2,2} &= -\frac{\partial}{\partial \rho} \left(\rho \cos \theta g_{2,1} + \frac{\rho^2 \sin^2 \theta}{v} g_{2,0} \right) + \frac{\partial}{\partial \theta} (\sin \theta g_{2,1}) \\ &\quad - 2nav \delta(\vec{r} - \vec{R}_1) f_{1,2}^{(B)} + n \delta(\vec{r} - \vec{R}_1) \Gamma(\rho, F_{1,2}^{(B)}). \end{aligned} \quad (46)$$

We notice that the equations thus obtained are linear inhomogeneous differential equations of the form $Lg_{2,j} = b_j$ ($j=0, 1, 2$), where $L = [\vec{v} \cdot \vec{\nabla}_{\vec{r}} + v(\partial/\partial\rho) + 2nav]$ is a linear differential operator and b_j for $j=0, 1, 2$ are the inhomogeneous terms on the r.h.s. of Eqs. (44), (45) and (46) respectively.

We will need to solve Eqs. (44)–(46), in conjunction with the equations for $G_{2,0}$, $G_{2,1}$ and $G_{2,2}$ obtained by directly integrating Eqs. (44)–(46) over ρ from 0 to ∞ . The equations for the corresponding $G_{2,i}$ are then

$$\left[\vec{v} \cdot \vec{\nabla}_{\vec{r}} - n \int d\vec{R}_2 \bar{T}_{-,2} \right] G_{2,0} = n \bar{T}_{-,1} F_{1,0}^{(B)}, \quad (47)$$

$$\left[\vec{v} \cdot \vec{\nabla}_{\vec{r}} - n \int d\vec{R}_2 \bar{T}_{-,2} \right] G_{2,1} = \frac{\partial}{\partial \theta} (\sin \theta G_{2,0}) + n \bar{T}_{-,1} F_{1,1}^{(B)}, \quad (48)$$

and

$$\left[\vec{v} \cdot \vec{\nabla}_{\vec{r}} - n \int d\vec{R}_2 \bar{T}_{-,2} \right] G_{2,2} = \frac{\partial}{\partial \theta} (\sin \theta G_{2,1}) + n \bar{T}_{-,1} F_{1,2}^{(B)}. \quad (49)$$

The equations for $G_{2,0}$, $G_{2,1}$ and $G_{2,2}$ are also linear differential equations of the form $L'G_{2,j}=B_j$, where $L'=[\vec{v}\cdot\vec{\nabla}_{\vec{r}}-n\int d\vec{R}_2\vec{T}_{-,2}]$ and B_j , for $j=0,1,2$, are the inhomogeneous terms on the r.h.s. of Eqs. (47), (48) and (49) respectively.

To solve these equations, we take Fourier transforms of g_2 and of G_2 in the variables \vec{r} and in the velocity angle, θ . That is, we define $\tilde{g}_2(\vec{k})=(1/\sqrt{V})\int_V d\vec{r} g_2 e^{-i\vec{k}\cdot(\vec{r}-\vec{R}_1)}$ and calculate $\tilde{g}_{2,0}(\vec{k})$, $\tilde{g}_{2,1}(\vec{k})$ and $\tilde{g}_{2,2}(\vec{k})$ in the \vec{k} -basis, using periodic boundary conditions. Similarly, we define the m -th angular mode of $\tilde{g}_2(\vec{k})$ as $\tilde{g}_2^{(m)}(\vec{k})=(1/\sqrt{2\pi})\int d\theta e^{-im\theta}\tilde{g}_2(\vec{k})$. We also define $\tilde{G}_2^{(m)}(\vec{k})$ in an analogous way. Thus, corresponding to Eqs. (44)–(46) and Eqs. (47)–(49), we have two sets of three equations to be solved, one involving $\tilde{g}_{2,0}(\vec{k})$, $\tilde{g}_{2,1}(\vec{k})$ and $\tilde{g}_{2,2}(\vec{k})$, and the other involving $\tilde{G}_{2,0}(\vec{k})$, $\tilde{G}_{2,1}(\vec{k})$ and $\tilde{G}_{2,2}(\vec{k})$, in (\vec{k}, m) basis. In this basis, the operators $L_{\vec{k}}=[i\vec{k}\cdot\vec{v}+v(\partial/\partial\rho)+2nav]$ and $L'_{\vec{k}}=[i\vec{k}\cdot\vec{v}-n\int d\vec{R}_2\vec{T}_{-,2}]$ are both infinite dimensional matrices in m -space and both of them have nonzero off-diagonal elements due to the term $i\vec{k}\cdot\vec{v}$ generated from the operator $\vec{v}\cdot\vec{\nabla}_{\vec{r}}$. However, it is easily seen that these off-diagonal elements are proportional to $\delta_{m,m+1}$ and $\delta_{m,m-1}$ and they are easily treated.

A further simplification can be made by noticing that the schematic forms of the solutions are $\tilde{g}_{2,j}(\vec{k})=[L_{\vec{k}}]^{-1}b_j(\vec{k})$ and $\tilde{G}_{2,j}(\vec{k})=[L'_{\vec{k}}]^{-1}B_j(\vec{k})$ and hence the dominant parts of $\tilde{g}_{2,j}(\vec{k})$ and $\tilde{G}_{2,j}(\vec{k})$ will come, loosely speaking, from the eigenfunctions of $L_{\vec{k}}$ and $L'_{\vec{k}}$ having the smallest eigenvalues. The lowest eigenvalues of $L'_{\vec{k}}$ are $\propto k^2$ due to the contributions from the hydrodynamic modes.⁽¹¹⁾ Thus, to capture the dominant part of the solutions, we should solve the equations in the range $k=|\vec{k}|\ll l^{-1}$, the inverse mean free path, and use perturbation expansions in the small parameter kl . We will not, in our analysis, follow the mode expansion technique, as it is simpler to calculate G_2 directly. However, one can use mode expansions and one finds that the results of both the methods agree.

4.1. Solution for G_2

To solve for G_2 first we need to know the solutions of the Lorentz–Boltzmann equation for $F_{1,0}^{(B)}$, $F_{1,1}^{(B)}$ and $F_{1,2}^{(B)}$. These are given by⁽⁸⁾

$$F_{1,0}^{(B)}=\frac{1}{2\pi}, \quad F_{1,1}^{(B)}=\frac{3}{16\pi nav}\cos\theta, \quad \text{and} \quad F_{1,2}^{(B)}=\frac{45}{512\pi(nav)^2}\cos 2\theta. \quad (50)$$

We note that in m -space, defined above, the m -th diagonal element of the infinite matrix $L'_{\vec{k}}$ is $4m^2v/[l(4m^2-1)]$ while the off-diagonal elements

are $ikv \delta_{m,m\pm 1}$. Thus an expansion in $\vec{k} = kl$ can be easily obtained by considering successively larger parts of the matrix L'_k in the index m , starting with 3×3 , 5×5 matrices and so on, chosen in such a way that the element of L'_k corresponding to $m=0$ appears as the center element of these matrices. As we want to make our results correct up to $O(\vec{k}^0)$, we need to increase the size of these matrices till the expressions of $\tilde{G}_{2,j}(\vec{k})$ obtained from $\tilde{G}_{2,j}(\vec{k}) = [L'_k]^{-1} B_j(\vec{k})$ (for $j=0, 1, 2$) converges up to $O(\vec{k}^0)$. Also, as we want to obtain the expression of λ_+ and λ_- in the leading field-dependent order, which is ε^2 ; we need the solutions of all the m -modes of $\tilde{G}_{2,0}(\vec{k})$, $\tilde{G}_{2,1}(\vec{k})$ and $\tilde{G}_{2,2}(\vec{k})$ that are necessary to obtain all the terms of $f_1^{(R)}$ that are $\propto \varepsilon^2$ and contribute to this leading field-dependent order of λ_+ and λ_- . In more explicit form, this means that we definitely need the solutions of $\tilde{G}_{2,0}^{(m=0)}(\vec{k})$, $\tilde{G}_{2,1}^{(m=0)}(\vec{k})$, $\tilde{G}_{2,1}^{(m=\pm 1)}(\vec{k})$, $\tilde{G}_{2,1}^{(m=\pm 2)}(\vec{k})$ and $\tilde{G}_{2,2}^{(m=0)}(\vec{k})$ up to $O(\vec{k}^0)$. However, once we present these solutions, from the structure and properties of them, it will turn out that we will also need the expressions of $\tilde{G}_{2,2j}^{(m=0)}(\vec{k})$ in the leading order of \vec{k} for $j=2, 3, \dots$, to consistently obtain all the terms, that are $\propto \varepsilon^2$.

At the ε^0 or equilibrium order, we find

$$\tilde{G}_{2,0}^{(m)}(\vec{k}) = 0 \quad \forall m. \tag{51}$$

Proceeding to order ε , we find that $\tilde{G}_{2,1}^{(m)}(\vec{k})$'s obey

$$\begin{aligned} \frac{iv}{2} [(k_x + ik_y) \tilde{G}_{2,1}^{(m+1)}(\vec{k}) + (k_x - ik_y) \tilde{G}_{2,1}^{(m-1)}(\vec{k})] + \frac{8navm^2}{4m^2 - 1} \tilde{G}_{2,1}^{(m)}(\vec{k}) \\ = -\frac{1}{2\sqrt{2\pi V}} (\delta_{m,1} + \delta_{m,-1}), \end{aligned} \tag{52}$$

where it turns out that we need a 5×5 matrix block corresponding to $m = -2, -1, 0, 1,$ and 2 to get the solutions of $\tilde{G}_{2,1}^{(m)}(\vec{k})$ up to $O(k^0)$ that are relevant for us, yielding

$$\begin{aligned} \tilde{G}_{2,1}^{(m=0)}(\vec{k}) &= \frac{ik_x}{vk^2 \sqrt{2\pi V}}, \\ \tilde{G}_{2,1}^{(m=1)}(\vec{k}) &= -\frac{3ik_y(k_x - ik_y)}{16navk^2 \sqrt{2\pi V}}, & \tilde{G}_{2,1}^{(m=-1)}(\vec{k}) &= \frac{3ik_y(k_x + ik_y)}{16navk^2 \sqrt{2\pi V}}, \\ \tilde{G}_{2,1}^{(m=2)}(\vec{k}) &= -\frac{45k_y(k_x - ik_y)^2}{1024(na)^2 vk^2 \sqrt{2\pi V}}, \\ \text{and } \tilde{G}_{2,1}^{(m=-2)}(\vec{k}) &= \frac{45k_y(k_x + ik_y)^2}{1024(na)^2 vk^2 \sqrt{2\pi V}}. \end{aligned} \tag{53}$$

Notice that we have also calculated $\tilde{G}_{2,1}^{(m=-2)}(\vec{k})$ and $\tilde{G}_{2,1}^{(m=2)}(\vec{k})$, even though they are $O(k)$, because they affect the $O(k^0)$ solution for $\tilde{G}_{2,2}^{(m=0)}(\vec{k})$.

For order ε^2 , the relevant $\tilde{G}_{2,2}^{(m)}(\vec{k})$'s are then calculated using Eq. (53) and considering a 5×5 matrix block of $L_{\vec{k}}^L$. There we need only the solution for $\tilde{G}_{2,2}^{(m=0)}(\vec{k})$:

$$\tilde{G}_{2,2}^{(m=0)}(\vec{k}) = \frac{k_x^2}{v^2 k^4 \sqrt{2\pi V}} + \frac{45(2k_x^2 - 5k_y^2)}{1024(nav)^2 k^2 \sqrt{2\pi V}}. \tag{54}$$

Examining the properties of the solutions, Eqs. (53)–(54) and observing from Eqs. (47)–(49) the way the solution of $G_{2,j}$ affects the solution of $G_{2,(j+1)}$, one sees that the leading power of k in the expression of $\tilde{G}_{2,2j}^{(m=0)}(\vec{k})$ for $(j = 1, 2, 3, \dots)$ is k^{-2j} . However, in the expression of G_2 , $\tilde{G}_{2,2j}^{(m=0)}(\vec{k})$ appears with a factor of ε^{2j} . When G_2 is finally calculated, after a summation of the appropriate \vec{k} -values,³ the contribution of the sum of all the effects coming from the $O(k^{-2j})$ terms of the $\tilde{G}_{2,2j}^{(m=0)}(\vec{k})$'s is seen to be in the same order of density of scatterers as the $O(k^0)$ term on the r.h.s. of Eq. (54). In fact, it also turns out that the $O(k^{-2j})$ terms of the $\tilde{G}_{2,2j}^{(m=0)}(\vec{k})$'s are the only ones among the $\tilde{G}_{2,j}^{(m)}(\vec{k})$'s that contribute to $f_1^{(R)}$ in the order of ε^2 . This implies that along with the solutions, Eqs. (51), (53), and (54), we also need to include the $O(k^{-2j})$ term of $\tilde{G}_{2,2j}^{(m=0)}(\vec{k})$ to be consistent. If one just considers this $O(k^{-2j})$ term in $\tilde{G}_{2,2j}^{(m=0)}(\vec{k})$, then it is easy to see that they satisfy a recurrence relation for $j \geq 1$:

$$\tilde{G}_{2,2(j+1)}^{(m=0)}(\vec{k}) = -\frac{k_x^2}{v^2 k^4} \tilde{G}_{2,2j}^{(m=0)}(\vec{k}), \tag{55}$$

i.e.,

$$\tilde{G}_{2,2j}^{(m=0)}(\vec{k}) = \frac{(-1)^{j-1}}{\sqrt{2\pi V}} \left(\frac{k_x^2}{v^2 k^4} \right)^j. \tag{56}$$

The solutions, Eqs. (51), (53), (54) and (56), are then used to determine the integration constants that arise when we solve the differential Eqs. (44)–(46). It is important to note that the first term on the right hand side of Eq. (54) is inversely proportional to k^2 . This is the origin of the logarithmic terms we find below.

4.2. Solution for g_2

Here we apply the same procedure to solve for the $\tilde{g}_2(\vec{k})$'s from the equations $L_{\vec{k}} \tilde{g}_2(\vec{k}) = b_j(\vec{k})$ for $j = 0, 1, 2$. This time, the elements of $L_{\vec{k}}$ are

³ To see how the \vec{k} -integration is performed, see the last paragraph of Section 4.2.

differential operators in the variable ρ and the corresponding constants of integrations are determined using the solutions of $\tilde{G}_2^{(m)}(\vec{k})$'s while maintaining that $\tilde{g}_2^{(m)}(\vec{k})$'s go to zero as $\rho \rightarrow 0$ and as $\rho \rightarrow \infty$. We also note that for our purpose, solutions of the $\tilde{g}_2(\vec{k})$'s are only needed for $\rho > a/2$ as the solution of the $\tilde{g}_2(\vec{k})$'s for $\rho < a/2$ gives rise to higher order density corrections than under consideration here. These solutions can also be obtained by the mode expansion technique discussed above in the paragraph preceding Section 4.1. However, as it is fairly straightforward to solve the differential Eqs. (44)–(46) for $\rho > a/2$, we directly write down the necessary solutions up to $O(k^0)$.

We obtain, for $\rho > a/2$,

$$\tilde{g}_{2,0}^{(m)}(\vec{k}) = \frac{2na}{\sqrt{2\pi V}} (1 - 2na\rho) e^{-2na\rho},$$

$$\begin{aligned} \tilde{g}_{2,1}^{(m=-1)}(\vec{k}) = & \frac{1}{v\sqrt{2\pi V}} \left[-\frac{k_x(k_x + ik_y)}{8k^2} + \frac{k_x(k_x + ik_y)}{2k^2} 2na\rho \right. \\ & \left. + \frac{2na\rho}{8} + \frac{(2na\rho)^2}{2} - \frac{(2na\rho)^3}{4} \right] e^{-2na\rho}, \end{aligned}$$

$$\tilde{g}_{2,1}^{(m=0)}(\vec{k}) = \frac{ik_x}{vk^2\sqrt{2\pi V}} 2nae^{-2na\rho},$$

$$\begin{aligned} \tilde{g}_{2,1}^{(m=1)}(\vec{k}) = & \frac{1}{v\sqrt{2\pi V}} \left[-\frac{k_x(k_x - ik_y)}{8k^2} + \frac{k_x(k_x - ik_y)}{2k^2} 2na\rho \right. \\ & \left. + \frac{2na\rho}{8} + \frac{(2na\rho)^2}{2} - \frac{(2na\rho)^3}{4} \right] e^{-2na\rho}, \end{aligned}$$

$$\begin{aligned} \tilde{g}_{2,2}^{(m=0)}(\vec{k}) = & \frac{k_x^2}{v^2k^4\sqrt{2\pi V}} 2nae^{-2na\rho} \\ & + \frac{1}{2v^2\sqrt{2\pi V}} \left[\frac{45}{128na} - \frac{315k_y^2}{256nak^2} + \frac{k_x^2}{4k^2}\rho - \frac{11}{8} 2na\rho^2 \right. \\ & - \frac{13}{8} \frac{k_x^2}{k^2} 2na\rho^2 + \frac{19}{24} (2na)^2 \rho^3 + \frac{k_x^2}{2k^2} (2na)^2 \rho^3 \\ & \left. + \frac{13}{24} (2na)^3 \rho^4 - \frac{1}{8} (2na)^4 \rho^5 \right] e^{-2na\rho}, \end{aligned} \tag{57}$$

and for the $O(k^{-2j})$ terms in $\tilde{G}_{2,2j}^{(m=0)}(\vec{k})$ we have

$$\tilde{g}_{2,2j}^{(m=0)}(\vec{k}) = \frac{(-1)^{j-1}}{\sqrt{2\pi V}} \left(\frac{k_x^2}{v^2 k^4} \right)^j 2nae^{-2nap}. \quad (58)$$

We point out that all of the terms in each of the square brackets, in each of the above three equations, are of the same order in the density. This can be seen easily by noting that ρ is typically of order $(2na)^{-1}$, so that $(2nap)$ is typically independent of the density. The solutions, Eqs. (51), (53), (54), (56), (57) and (58) now can be assembled to calculate G_2 and g_2 in (\vec{r}, \vec{v}) and (\vec{r}, \vec{v}, ρ) space respectively and feed the results into the r.h.s. of Eqs. (31) and (36) to obtain $f_1^{(R)}$. This involves a summation of different m and \vec{k} -values. In the infinite volume limit the \vec{k} -sum can be converted to an integration over \vec{k} . The sum over m is straightforward, but we have to remember that the integration over \vec{k} has to be carried out in a range $k \leq k_0 \sim l^{-1}$. Secondly, since we have expanded the distribution functions in powers of ε and then subsequently in powers of k , the lower limit of k for the \vec{k} -integration cannot be taken to be zero. To determine this lower limit of k for the k -integration, we observe that the expansion in ε cannot be carried out for those values of k where $k < \varepsilon/(2v)$, so that the value $\varepsilon/(2v)$ forms a natural lower cut-off for the Fourier transform. Our solutions of G_2 and g_2 therefore do not hold for $k < \varepsilon/(2v)$ and to do a satisfactory perturbation theory in the range $k < \varepsilon/(2v)$, one needs to consider both the ε and \vec{k} -dependent terms together. After doing so, one finds that such a perturbation theory does not affect our results at the present density order.⁽¹⁸⁾

Before performing the integration over \vec{k} , we notice that in two dimensions, the numerator of the \vec{k} -integral is proportional to $k dk$. This means that any part of the solutions of $\tilde{g}_2(\vec{k})$ or $\tilde{G}_2(\vec{k})$ having a leading power of k of order 2 or higher in the denominator gives rise to a singularity at $k \rightarrow 0$ for the \vec{k} -integral. First, the highest leading power of k in the denominators of Eqs. (51), (53), (54) and (57) is k^2 , occurring in $\tilde{G}_{2,2}^{(m=0)}(\vec{k})$ and $\tilde{g}_{2,2}^{(m=0)}(\vec{k})$ respectively. These terms proportional to k^{-2} give rise to a logarithmic electric field dependence once the \vec{k} -integration is performed for $\varepsilon/(2v) \leq k \leq k_0$. The rest of the terms in these solutions supply only analytic field dependences that can be expressed as power series in ε . Secondly, even though the solutions given in Eqs. (56) and (58) have higher powers of k than k^2 in the denominators, they also come with subsequently higher powers of ε in their numerators. Thus, when the \vec{k} -integration is performed, they contribute terms proportional to ε^2 or higher, to g_2 or G_2 . Consequently, in addition to analytic field dependent terms, in our present

approximation we have only one nonanalytic field dependent term appearing in g_2 and G_2 and that is proportional to $\tilde{\varepsilon}^2 \ln \tilde{\varepsilon}$. No doubt there exist further nonanalytic terms in higher orders in $\tilde{\varepsilon}$, but their calculation would require a careful consideration of various terms we have neglected here, such as the repeated ring contributions.

4.3. Solution for $f_1^{(R)}$ and the Calculation of $\lambda_+^{(R)}$

Once the solutions, Eqs. (51), (53), (54), (56), (57) and (58) are inserted in Eqs. (31) and (37) and the \vec{k} -integration is performed in the range $\varepsilon/(2v) \leq k \leq k_0$, we get, by the method described in Eqs. (40)–(43), the following equations to be solved to obtain $f_1^{(R)}$ and $F_1^{(R)}$ respectively

$$\begin{aligned}
 & -\varepsilon \frac{\partial}{\partial \theta} (\sin \theta f_1^{(R)}) + \frac{\partial}{\partial \rho} \left\{ \left(v + \rho \varepsilon \cos \theta + \frac{\rho^2 \varepsilon^2 \sin^2 \theta}{v} \right) f_1^{(R)} \right\} + 2nav f_1^{(R)} \\
 & = \Theta \left(\frac{a}{2} - \rho \right) \frac{4v\rho}{a \sqrt{1 - \left(\frac{2\rho}{a} \right)^2}} \\
 & \quad \times \int_{-\pi/2}^{\pi/2} d\phi \cos \phi b_\sigma \left[\frac{\varepsilon^2}{8\pi^2 v^2} \left\{ \ln \frac{2vk_0}{\varepsilon} \right\} \right. \\
 & \quad - \frac{k_0^2}{8\pi^2} \left\{ \frac{3}{16nav} \varepsilon \cos \theta + \frac{135}{2048(nav)^2} \varepsilon^2 \right\} + \frac{A\varepsilon^2}{16\pi^3 v^2} \Big] \\
 & \quad - 2av \left[\frac{\varepsilon^2}{8\pi^2 v^2} \left\{ \ln \frac{2vk_0}{\varepsilon} \right\} 2nae^{-2nap} \right. \\
 & \quad + \frac{k_0^2}{8\pi^2} e^{-2nap} \left\{ 2na(1 - 2nap) \right. \\
 & \quad - \frac{\varepsilon}{v} \left[\frac{(2nap)^3}{2} - (2nap)^2 - \frac{3}{4} 2nap + \frac{1}{8} \right] \cos \theta \\
 & \quad - \frac{\varepsilon^2}{4nav^2} \left[\frac{(2nap)^5}{8} - \frac{13}{24} (2nap)^4 - \frac{25}{24} (2nap)^3 \right. \\
 & \quad \left. \left. + \frac{35}{16} (2nap)^2 - \frac{2nap}{8} + \frac{135}{256} \right] \right\} \\
 & \quad \left. + \frac{1}{16\pi^3 v^2} A\varepsilon^2 2nae^{-2nap} \right] + \dots, \tag{59}
 \end{aligned}$$

and

$$\begin{aligned}
 & -\varepsilon \frac{\partial}{\partial \theta} (\sin \theta F_1^{(R)}) - nav \int_{-\pi/2}^{\pi/2} d\phi \cos \phi (b_\sigma - 1) F_1^{(R)} \\
 & = \int_{-\pi/2}^{\pi/2} d\phi \cos \phi (b_\sigma - 1) \left[\frac{a\varepsilon^2}{8\pi^2 v} \ln \left(\frac{2vk_0}{\varepsilon} \right) \right. \\
 & \quad \left. - \frac{avk_0^2}{8\pi^2} \left\{ \frac{3}{16nav} \varepsilon \cos \theta + \frac{135}{2048(nav)^2} \varepsilon^2 \right\} \right. \\
 & \quad \left. + \frac{a}{16\pi^3 v} A\varepsilon^2 + \dots \right] \\
 & = \frac{ak_0^2}{16\pi^2 na} \varepsilon \cos \theta + \dots, \tag{60}
 \end{aligned}$$

where b_σ has been defined in Eq. (33). The A -dependent terms in Eqs. (59) and (60) originate from Eqs. (58) and (56) respectively after the \vec{k} -integration is carried out. Here A is the integral⁴

$$A = \int_0^{2\pi} d\phi \cos^2 \phi \ln [1 + \frac{1}{4} \cos^2 \phi] = 0.53536\dots \tag{61}$$

The dominant effect of the ring term on the single particle distribution function, i.e., $f_1^{(R)}$, can now be determined from Eqs. (59) and (60). It is also of some interest to give a crude estimate of the terms that we have neglected. One knows from other studies in the kinetic theory of gases^(10, 22) that excluded volume corrections to Boltzmann equation results are the numerically most important corrections, until the density of the system becomes high enough that the mean free path of a particle is less than the size of the particle itself. These excluded volume corrections are provided by the Enskog theory, and this theory can be applied to the Lorentz gas, as well.⁽²⁰⁾ In our case, the Enskog corrections can be included by replacing the density parameter n by $n(1 - \pi na^2)^{-1} \approx n(1 + \pi na^2)$ in the Boltzmann equation. The Enskog correction affects both λ_0 and the ε -dependent terms in the expressions for λ_\pm in Eq. (14). Along with the Enskog correction there are other correction terms that affect both λ_0 and the field-dependent terms λ_\pm ^(18, 23) at the same density order as the Enskog correction. Also, the terms that have been dropped to obtain Eq. (16) from Eq. (11), contribute to λ_\pm at the same density order as the Enskog correction. However, since the principal objective of this paper is to investigate the nonanalytic

⁴ We thank the referee for pointing out an error in a previous calculation of this integral.

contribution of the ring term to the Lyapunov exponents, we will ignore the Enskog and related corrections from our consideration. Thus, using Eqs. (59) and (60), one can express the full solutions of f_1 and F_1 as sums of a solution in the Boltzmann regime, a correction due to the ring term and a correction due to the Enskog term, plus all of the other terms we have neglected, as

$$f_1 = f_1^{(B)} + f_1^{(R)} + \dots, \tag{62}$$

$$F_1 = F_1^{(B)} + F_1^{(R)} + \dots. \tag{63}$$

Consequently, for the positive Lyapunov exponent λ_+ we have,

$$\lambda_+ = \lambda_+^{(B)} + \lambda_+^{(R)} + \dots. \tag{64}$$

The solution of $F_1^{(R)}$ is quite straightforward,

$$F_1^{(R)} = \frac{3ak_0^2}{128\pi^2(na)^2 v} \varepsilon \cos \theta + \dots. \tag{65}$$

However, to solve for $f_1^{(R)}$ we find that in addition to the analytic field-dependent terms which can be expressed as a power series in ε , there is a non-analytic field-dependent term in $f_1^{(R)}$ proportional to $\tilde{\varepsilon}^2 \ln \tilde{\varepsilon}$. Thus, with

$$f_1^{(R)} = f_{1, \text{analytic}}^{(R)} + f_{1, \text{non-analytic}}^{(R)} \tag{66}$$

we have

$$\begin{aligned} & f_{1, \text{analytic}}^{(R)} \\ &= -\frac{ak_0^2}{4\pi^2} \left[2na\rho - \frac{(2na\rho)^2}{2} - \frac{\varepsilon \cos \theta}{2nav} \left\{ \frac{(2na\rho)^4}{4} - (2na\rho)^3 + \frac{(2na\rho)^2}{8} + \frac{2na\rho}{8} \right\} \right. \\ & \quad + \frac{\varepsilon^2}{4(nav)^2} \left\{ -\frac{(2na\rho)^6}{32} + \frac{11}{48} (2na\rho)^5 - \frac{(2na\rho)^4}{96} - \frac{79}{96} (2na\rho)^3 \right. \\ & \quad \left. \left. + \frac{3}{32} (2na\rho)^2 - \frac{135}{512} (2na\rho) + \frac{135}{512} \right\} \right] e^{-2na\rho} \\ & \quad + \frac{a\varepsilon^2}{(2\pi)^3 v^2} A(1 - 2na\rho) e^{-2na\rho} + \dots, \quad \text{for } \rho > \frac{a}{2} \\ &= \left[\frac{aA}{(2\pi)^3 v^2} - \frac{135ak_0^2}{512(4\pi nav)^2} \right] \left\{ 1 - \sqrt{\left(\frac{2\rho}{a}\right)^2} \right\} \varepsilon^2 + \dots, \quad \text{for } \rho < \frac{a}{2}, \tag{67} \end{aligned}$$

and

$$\begin{aligned}
 f_{1, \text{non-analytic}}^{(\text{R})} &= \frac{a\varepsilon^2}{4\pi^2 v^2} \left\{ \ln \frac{2vk_0}{\varepsilon} \right\} (1 - 2na\rho) e^{-2na\rho}, & \text{for } \rho > \frac{a}{2}, \\
 &= \frac{a\varepsilon^2}{4\pi^2 v^2} \left\{ \ln \frac{2vk_0}{\varepsilon} \right\} \left[1 - \sqrt{\left(\frac{2\rho}{a}\right)^2} \right], & \text{for } \rho < \frac{a}{2}. \quad (68)
 \end{aligned}$$

Notice that the ring contribution to the distribution function in Eqs. (67) and (68) satisfies the boundary conditions that $f_1^{(\text{R})} \rightarrow 0$ as $\rho \rightarrow 0$ and $\rho \rightarrow \infty$. Equations (67) and (68) also satisfy continuity at $\rho = a/2$ at the leading density order. The distribution functions, Eqs. (65), (67) and (68), are all the ones that we need to calculate $\lambda_+^{(\text{R})}$. Consequently,

$$\lambda_+^{(\text{R})} = \lambda_{+, \text{analytic}}^{(\text{R})} + \lambda_{+, \text{non-analytic}}^{(\text{R})} \quad (69)$$

and using the the definition of Lyapunov exponents in Eq. (12), we have

$$\begin{aligned}
 \lambda_{+, \text{analytic}}^{(\text{R})} &= \int_0^{2\pi} d\theta \int_{a/2}^{\infty} d\rho \frac{f_{1, \text{analytic}}^{(\text{R})}}{\rho} \\
 &= -\frac{ak_0^2 v}{4\pi} - \frac{ak_0^2 l^2 \varepsilon^2}{2\pi v} \left\{ \frac{13}{96} - \frac{135}{512} (\ln 2na^2 + \mathcal{C}) \right\} \\
 &\quad - 0.53536 \frac{a\varepsilon^2}{(2\pi)^2 v} (\ln 2na^2 + \mathcal{C}) + \dots, \quad (70)
 \end{aligned}$$

and

$$\begin{aligned}
 \lambda_{+, \text{non-analytic}}^{(\text{R})} &= \int_0^{2\pi} d\theta \int_{a/2}^{\infty} d\rho \frac{f_{1, \text{non-analytic}}^{(\text{R})}}{\rho} \\
 &= -\frac{a\varepsilon^2}{2\pi v} \left\{ \ln \frac{2k_0 v}{\varepsilon} \right\} (\ln 2na^2 + \mathcal{C}), \quad (71)
 \end{aligned}$$

where l is the mean free path and \mathcal{C} is Euler's constant, $\mathcal{C} = 0.5772\dots$

4.4. Calculation of $\lambda_-^{(\text{R})}$

To calculate the corresponding effect of the ring term on λ_- , we make use of the relation Eq. (13). It is easy to calculate the effect of the ring term on $\langle \alpha \rangle_{\text{NESS}}$ using $F_1^{(\text{R})}$ already determined in the previous section. Thus, using

$$\lambda_+ + \lambda_- = -\langle \alpha \rangle_{\text{NESS}}, \quad (72)$$

and a complete analogy to Eqs. (62)–(64), we can calculate the terms of $\langle \alpha \rangle_{\text{NESS}}$:

$$\langle \alpha \rangle_{\text{NESS}} = \langle \alpha \rangle_{\text{NESS}}^{(\text{B})} + \langle \alpha \rangle_{\text{NESS}}^{(\text{R})} + \dots, \quad (73)$$

with

$$\langle \alpha \rangle_{\text{NESS}}^{(\text{R})} = \frac{3ak_0^2 l^2 \varepsilon^2}{32\pi v} + \dots. \quad (74)$$

Following Eqs. (62–64), we now express λ_- as $\lambda_- = \lambda_-^{(\text{B})} + \lambda_-^{(\text{R})} + \dots$, satisfying $\lambda_+^{(\text{R})} + \lambda_-^{(\text{R})} = -\langle \alpha \rangle_{\text{NESS}}^{(\text{R})}$. This leads us to

$$\begin{aligned} \lambda_{-, \text{analytic}}^{(\text{R})} &= \frac{ak_0^2 v}{4\pi} - \frac{ak_0^2 l^2 \varepsilon^2}{2\pi v} \left\{ \frac{5}{96} + \frac{135}{512} (\ln 2na^2 + \mathcal{C}) \right\} \\ &+ 0.53536 \frac{a\varepsilon^2}{(2\pi)^2 v} (\ln 2na^2 + \mathcal{C}) + \dots, \end{aligned} \quad (75)$$

and

$$\lambda_{-, \text{non-analytic}}^{(\text{R})} = \frac{a\varepsilon^2}{2\pi v} \left\{ \ln \frac{2k_0 v}{\varepsilon} \right\} (\ln 2na^2 + \mathcal{C}), \quad (76)$$

where l is the mean free path and \mathcal{C} is Euler’s constant, $\mathcal{C} = 0.5772\dots$

5. THE FIELD-DEPENDENT COLLISION FREQUENCY AND ITS EFFECTS ON THE LYAPUNOV EXPONENTS

As stated before, our second main purpose was the derivation of the leading non-analyticity in the field dependence of the Lyapunov exponents. In analogy with the transport coefficients, we expected these non-analyticities to result from the long time behavior of the ring terms, which we found confirmed in the preceding section. Some further thought reveals we can estimate the non-analytic field dependence in a simple way.

In the presence of a thermostatted field there are two types of contributions to the positive Lyapunov exponent of the two-dimensional Lorentz gas:

- (1) contributions from the bending of the trajectories by the fields and
- (2) contributions from the divergence of trajectory pairs at collisions.

The first type of contributions are of order $\tilde{\varepsilon}^2$ in the Boltzmann approximation. We expect that the coefficient of this term will pick up higher density corrections and there will be additional terms of higher orders in $\tilde{\varepsilon}$. But we have not found any indications for corrections of lower order than $\tilde{\varepsilon}^2$ resulting from the field-bending contributions.

The collisional contributions can be generally expressed as an average of the form $\nu \langle \ln(|\delta\vec{v}'|/|\delta\vec{v}|) \rangle_c$, with $\delta\vec{v}'$ and $\delta\vec{v}$ the velocity differences between the adjacent trajectories just after and just before a collision, respectively, ν the average collision frequency, and the angular brackets, $\langle \rangle_c$, indicating an average over collisions. At low densities even correlated collisions happen at large distances, i.e., in the order of a mean free path length apart from each other. Therefore their distribution of collision angles and hence their contribution to the average $\langle \rangle_c$, to the leading order in density remains the same as for uncorrelated collisions. We should then expect that at low densities the main effect of the correlated collisions on the Lyapunov exponents should be due to a change of the collision frequency ν as a result of correlated collisions taking place in the presence of the field. If the latter changes from ν_0 to $\nu_0 + \delta\nu$, then Eq. (15) predicts a change of the positive Lyapunov exponent of magnitude

$$\delta\lambda_+ = -\delta\nu \left\{ \ln \frac{a\nu_0}{v} + \mathcal{C} \right\}. \quad (77)$$

To obtain this result we have used the fact that the equilibrium, low density Lyapunov exponent, Eq. (15) can be written in the form

$$\lambda_0 = \nu_0 \left\{ 1 - \mathcal{C} - \ln \frac{a\nu_0}{v} \right\}, \quad (78)$$

where $\nu_0 = 2nav$. In order to understand why and how the thermostatted field changes the collision frequency we first recall that in equilibrium the collision frequency can be obtained simply by using the uniformity of the equilibrium distribution for the point particle in available phase space, with the result that $\nu = 2nav/(1 - \pi na^2)$. One just has to consider the probability that the light particle during an infinitesimal time dt will hit one of the scatterers. On the other hand, at a time t after a given initial time, the probability for a collision may be considered to be a sum of three contributions: the collision frequency obtained by assuming that all collisions are uncorrelated and independent of each other, *plus* the probability for a recollision with a scatterer with which it has collided before, *minus* the reduction of the collision probability due to any collected knowledge of where no scatterers are present. In equilibrium the last two contributions

have to cancel, as we demonstrate in the Appendix. In the presence of a field, however, this cancellation does not occur. This can easily be understood in a qualitative way following the argument that the cancellation in equilibrium occurs because the probability for return to the boundary of a scatterer is exactly the same as that for return to the boundary of a region where a scatterer could be, but in fact is not present (a virtual scatterer). In the presence of a field, the average velocity of the point particle before collision with a real scatterer will be in the direction of the field, and after the collision the average velocity will be anti-parallel to the field. The field will then tend to turn the particle around and have it move back in the direction of the scatterer. This effect enhances the probability of a recollision in comparison to that for an isotropic distribution around the scatterer. In a “virtual collision,” in which the velocity does not change, the particle, on average, ends up downstream (i.e., in the direction of the applied field) from the virtual scatterer and its recollision probability is decreased compared to that for an isotropic distribution.

In the Appendix, a quantitative calculation is given based on the following two assumptions:

- (1) After the real or virtual collision the spatial distribution of the point particle becomes centered around a point at a distance of a diffusion length from the scatterer and
- (2) for long times this distribution can be found by solving the diffusion equation. The resulting expression for δv is

$$\delta v = \frac{a\varepsilon^2}{2\pi v} \ln \frac{v_0}{\varepsilon}. \quad (79)$$

A more formal, but equivalent, way to obtain this result is by extending the method described by Latz, van Beijeren and Dorfman⁽²⁶⁾ for the low density distribution of time of free flights of the moving particle to include the contribution from ring events, so as to apply to a system in a thermostatted electric field. The main idea is to solve a kinetic equation for $f(\vec{r}, \vec{v}, t, \tau)$, the distribution of particles at a phase point (\vec{r}, \vec{v}) at time t such that their last collision took place at a time τ earlier, i.e., at time $t - \tau$. It is then easy to argue that the distribution of free flight times is simply the derivative of this (“last collision”) distribution with respect to $t - \tau$. We can then obtain a NESS average of the time of free flight and thereby calculate the field dependent collision frequency $\nu(\varepsilon) = \nu_0 + \delta\nu$. Since we want to show that the origin of the non-analytic field dependence of both λ_+ and λ_- is rooted in the non-analytic field dependence of collision frequency $\delta\nu$, let us keep only the non-analytic field-dependent term as the leading term

of the expansion of δv in the density of scatterers and in the electric field strength and write

$$\delta v = \beta \varepsilon^2 \ln \left\{ \frac{2k_0 v}{\varepsilon} \right\} + \dots, \quad (80)$$

where the quantity β has to be determined from the NESS average of τ , during the effect of the ring term on the NESS distribution function $f(\vec{r}, \vec{v}, \tau)$ with k_0 of the order of $1/(v_0 v)$. To obtain this distribution function, we follow exactly the same procedure as outlined in Sections 4 and 5, but this time, with the variable τ instead of ρ . Notice that, this time, even though the equations for corresponding f_1 and g_2 's are different, due to the difference in the dynamical equations for $\dot{\rho}$ and $\dot{\tau}$ during free flights and at collisions, the equations involving F_1 and G_2 's remain the same. The source of the non-analytic field-dependent term will surface again exactly from the $O(k^{-2})$ term in Eq. (54). As far as this non-analytic field-dependent term is concerned, at the lowest order of density, the variables ρ and τ are identical up to a multiplicative factor v . Both grow linearly with time in between collisions and are set back to (for ρ , almost) zero at each collision with a scatterer. One then recovers the corresponding non-analytic part of the NESS distribution function,⁽²⁷⁾ analogous to Eq. (68),

$$f_{1, \text{non-analytic}}^{(R)}(\vec{v}, \tau) = \frac{a\varepsilon^2}{4\pi^2 v} \left\{ \ln \frac{2vk_0}{\varepsilon} \right\} (1 - 2nav\tau) e^{-2nav\tau} \quad \text{for } \tau > 0 \quad (81)$$

from which β can be obtained to be

$$\beta = \frac{a}{2\pi v}, \quad (82)$$

after which, one easily recovers the result of Eq. (79).

6. DISCUSSION

While much of this paper is quite technical, there are two main points that we would like to emphasize: (1) We have developed a method which allows an extension of the calculation of the Lyapunov exponents for a two-dimensional Lorentz gas to higher densities than is possible by means of the ELBE. (2) The logarithmic terms obtained here, while small, are indicators of similar logarithmic terms which are certain to appear when these calculations are extended to general two-dimensional gases, where all of the particles move.

The first point allows one to contemplate a general kinetic theory for the calculation of sums, at least, of all positive, or of all negative Lyapunov exponents. Such an approach was also indicated by Dorfman, Latz, and van Beijeren,⁽²⁴⁾ for the KS-entropy of a dilute gas in equilibrium, but the theory there has not yet been developed beyond the Boltzmann equation. The relevance of the second point can be seen if one realizes that the linear Navier–Stokes transport coefficients of a two-dimensional gas diverge because of long time tail effects, of the type discussed here.⁽¹⁰⁾ In the general gas case therefore the logarithmic terms in the positive and negative Lyapunov exponents will not cancel as they do here, because the transport coefficients themselves should diverge as $\ln \tilde{\varepsilon}$ as ε approaches zero. Thus the logarithmic terms obtained here should be seen as precursors of the more important logarithmic terms that will appear in the theory of two-dimensional gases.

It is worth noting that the $\tilde{\varepsilon}^2 \ln \tilde{\varepsilon}$ term results from a long range correlation in time between the moving particle and the scatterers that is present in both the pair correlation functions, G_2 , and g_2 , either of which is proportional to the square of the electric field strength and the inverse square of the wave number, at small wave numbers and fields. This dependence is not present in the Lorentz gas in equilibrium, of course, but similar collision frequency arguments to those given here suggest that non-analytic terms may be present in the ring contributions to the positive Lyapunov exponent for trajectories on the fractal repeller for an open Lorentz gas. In this case the inverse system size, L^{-1} , plays the role of $\varepsilon/(2v)$, the lower limit of k for the integration over \vec{k} and one would expect to find terms of order $L^{-2} \ln L$ in the ring term for this case. This point is currently under investigation.

Finally we mention that neither the non-analytic terms found here, nor the excluded volume corrections included in the Enskog terms are able to account for the field dependence of the Lyapunov exponents as observed in the computer simulations by Dellago and Posch.⁽⁸⁾ This is not unexpected since we have not been systematic in computing the density dependence of the coefficient of ε^2 , nor have we considered higher order terms in ε beyond order $\varepsilon^2 \ln \varepsilon$. All of the neglected terms are likely to be numerically more important than the ones we have kept. There is also no indication in the simulation data for the Lyapunov exponents of a clear presence of the interesting logarithmic term in the applied field. Such logarithmic terms are typically difficult to detect in simulation data, without a careful hunt for them.⁽²⁰⁾ However, it may be easier to check, by means of computer simulation, the existence of the $\tilde{\varepsilon}^2 \ln \tilde{\varepsilon}$ term in the collision frequency than in the Lyapunov exponents. In any case, we would like to emphasize that computer simulation studies of thermostatted systems provide very useful

ways to check a number of phenomena predicted by the kinetic theory of moderately dense gases.

APPENDIX. DERIVATION OF THE FIELD-DEPENDENT COLLISION FREQUENCY

To derive the field dependence of the collision frequency we first approximate the probability of a recollision at time t as

$$P^{\text{rec}}(t) = \frac{\nu}{2} \int_0^{2\pi} d\theta \int_0^\infty d\tau \int_{\vec{v} \cdot \hat{\sigma} > 0} d\hat{\sigma} |\vec{v} \cdot \hat{\sigma}| R(\tau, \theta, \sigma) b_\sigma F^{(B)}(\theta). \quad (\text{A1})$$

Here $F^{(B)}(\theta)$ describes the Boltzmann distribution for the velocity in the NESS. The function $R(\tau, \theta, \sigma)$ describes the probability density for return to the circumference of a given scatterer in a time τ just after colliding with this scatterer with scattering vector $\hat{\sigma}$ and post-collisional velocity described by θ (see Fig. 4). We have ignored a possible dependence of the collision frequency ν on \hat{v} , which would only play a role at higher orders in the density.

Similarly the reduction of the collision frequency at time t due to virtual recollisions can be estimated as

$$P^{\text{nc}}(t) = -\frac{\nu}{2} \int_0^{2\pi} d\theta \int_0^\infty d\tau \int_{\vec{v} \cdot \hat{\sigma} > 0} d\hat{\sigma} |\vec{v} \cdot \hat{\sigma}| R(\tau, \theta, \sigma) F^{(B)}(\theta) \quad (\text{A2})$$

In equilibrium $F^{(B)}(\theta)$ is independent of θ , so one sees immediately that both terms cancel, as they should. In the presence of a thermostatted

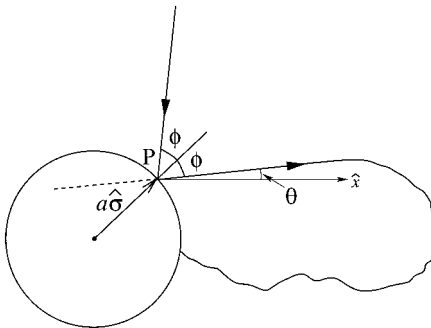


Fig. 4. A recollision taking place after a real (solid line) or corresponding virtual (dashed line) collision, followed by a post-collisional excursion maintaining on average the direction of velocity over a persistence length l_p .

field we need the explicit form of $F_1^{(B)}(\theta)$ up to the first field-dependent order, given in Eq. (50) as

$$F_1^{(B)}(\theta) = \frac{1}{2\pi} \left[1 + \frac{3\varepsilon}{8nav} \cos \theta \right] \quad (A3)$$

The function $R(\tau, \theta, \sigma)$ for large enough τ may be approximated by the product of $2av$ (velocity times cross section) and the probability density for finding the point particle at the position of the scatterer. For weak fields the latter may be approximated by the solution of a diffusion equation with a drift velocity $u\hat{x}$ in the $+x$ -direction and an initial density localized at the position $l_p\hat{\theta}$ with respect to the center of the scatterer. Here l_p is the persistence length, that is, the average distance traveled by a point particle in an equilibrium system in the direction of its initial velocity and $\hat{\theta}$ is the unit vector in the direction of the velocity right after the initial collision at $t-\tau$. The persistence length may be expressed as $l_p = \int_0^\infty dt \langle \hat{v} \cdot \vec{v}(t) \rangle$. Multiplying this by the constant speed v one finds with the aid of the Green-Kubo expression for the diffusion that $l_p = 2D/v$ in two dimensions. This assumption for the long time distribution may be understood by imagining that the first few free flights after the initial collision of the particle move it over a distance in the order of a mean free path in the direction of its initial postcollisional velocity before it starts to diffuse by virtue of further collisions with scatterers. Thus for large τ the distribution of the light particle will be centered around the point $l_p\hat{\theta}$ with respect to the center of the scatterer, and the final point, on the surface of the scatterer, may be approximated to be at the center of the scatterer as well, because of low density. These arguments lead to the explicit form for the recollision probability given by

$$R(\tau, \theta, \sigma) = 2av \frac{e^{-[l_p\hat{\theta} + u\tau\hat{x}]^2/(4D\tau)}}{4\pi D\tau}. \quad (A4)$$

Finally we need the explicit form $u = 3\varepsilon v/8v_0$ for the drift velocity to leading order in the density, and the identity that

$$\frac{1}{2} \int_{\vec{v} \cdot \hat{\sigma} > 0} d\hat{\sigma} |\vec{v} \cdot \hat{\sigma}| b_\sigma \cos \theta = -\frac{v}{3} \cos \theta. \quad (A5)$$

Then, after expanding

$$e^{-2l_p u \tau \hat{\theta} \cdot \hat{x}/(4D\tau)} = 1 - \frac{l_p u}{2D} \cos \theta + \dots, \quad (A6)$$

we can now do all the calculations needed to obtain the leading non-analytic term in the field expansion of the collision frequency. We find that

$$\delta v = avv \int_0^{2\pi} d\theta \int_0^\infty d\tau \frac{e^{-[I_p^2 + (u\tau)^2]/(4D\tau)}}{4\pi D\tau} \left[1 - \frac{l_p u}{2D} \cos \theta \right] \\ \times \int_{\vec{v} \cdot \hat{\sigma} > 0} d\hat{\sigma} |\vec{v} \cdot \hat{\sigma}| (b_\sigma - 1) \frac{3\epsilon \cos \theta}{16\pi nav}. \quad (A7)$$

After performing the integrations, we recover Eq. (79). Notice that the logarithm of $\tilde{\epsilon}$ results from the cut-off on the τ integration provided by the drift term in the exponential.

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